CONSTRUCTION AND ANALYSIS OF SEVERAL SERIES OF INCOMPLETE BLOCK DESIGNS

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Introduction

In agricultural field experiments or industrial experiments, the experimenter is often faced with the difficulty of testing a large number of varieties, all in the same experiment or he may be placed in a position of testing even a small number of treatments in smaller block sizes. In biological work on animals, for example, where the number of experimental units and their grouping into blocks is to some extent out of the control of the experimenter, it will be desirable, if at all possible to compare several treatments within block, but the size of the block will depend upon the particular species of animal and will often be such that it is impossible to include all treatments within a block. In such situation there are mainly two classes of designs, namely balanced incomplete block designs as introduced by Yates (1936 *a*, 1936 *b*, 1937) and partially balanced incomplete block designs as introduced by Bose and Nair (1939) which can be applied.

In recent days some problems in the construction of designs have been solved by combining several designs. Thus, Bose, Shrikhande and Parker (1960) disproved Euler's conjucture on orthogonal latin square of order $4n + 2$ by combining orthogonal arrays and balanced incomplete bolck designs with $\lambda = 1$. Again Das (1961) obtained rotatable designs by combining two factorial designs and also a factorial design and an incomplete block design. The present investigation was undertaken with a view to explore the possibilities of combining two incomplete block designs for obtaining new series of incomplete block designs. Several methods of obtaining incomplete block designs through this technique have been described below.

1. *Method of Construction of P.B.I.B. Design by Combining a P.B.I.B. Design and a B.I.B. Design*:

Let there be a P.B.I.B. design with 'm' associate classes and parameters (v, r, b, k, λ_1, λ_2, $\cdots$ λ_m, n_1, n_2, $\cdots$ n_m, $p_{jk}{}^i$). With the treatments present in any block of this design another B.I.B. design with parameter (k, k_1, r_1, b_1, λ) is formed. Thus in all there will be 'b' such designs formed out of the contents of the b blocks of the initial design. The final design formed by putting together all these bb_1 blocks each of size K_1, is a P.B.I.B. design with the same number of treatments and same number of associate classes. The parameters of this design will be (v, k_1, rr_1, bb_1, $\lambda\lambda_1$, $\cdots$ $\lambda\lambda_m$, n_1, n_2, $\cdots$ n_m, $p^i{}_{jk}$). In future, the original P.B.I.B. design will be called the primary design, the second design formed out of the contents of each of the blocks of the primary design will be called the secondary design, the design for each block having the same parameters, and the last design will be called the final design.

The proof that the final design is a P.B.I.B. design follows from the fact that since each of the treatments in a block in the primary design occurs r_1 times while forming the secondary designs and as the same treatment occurs in r blocks of the primary design, any treatment will occur rr_1 times in the final design. Similarly the number of blocks will be bb_1 and the number of times a pair of treatments (s-th associates) occur together in the same block is $\lambda\lambda_s$.

Analysis of these designs.—Though such designs can be analysed just like ordinary P.B.I.B. designs, there is also an alternative method of analysis of such designs. As it appears that the method will be interesting and may prove useful in some situations it has been presented below.

Treating each of the b secondary designs as distinct, we shall first analyse each of them separately. The estimates of k treatments obtained from any of the design, say the i-th one will then be substituted for the k treatments occurring in the i-th block of the primary design. We shall thus have b blocks of the primary design where the block contents are the estimates of the treatments obtained from the corresponding secondary designs. Next, treating such estimates as observations, an analysis of the primary design as a P.B.I.B. design will lead to the same estimates of the treatment effects as will be obtained from the analysis of the final design as ordinary P.B.I.B. design. Obtaining the estimates of the treatment effects in this way, the adjusted treatment S.S. can be obtained from ΣtQ where Q_i is the adjusted total for the i-th treatment obtained from the final design. The error S.S. can then be obtained by subtracting the adjusted treatment S.S. from the within block S.S. of the final design. The proof that these two methods of estimation of treatment estimates give the same result has been given below.

Denoting the treatments as $t_1, t_2, \cdots t_v$, the normal equations of the final design can be written as

$$\left(rr_1 - \frac{rr_1}{k_1}\right) t_i - \frac{\lambda\lambda_1}{k_1} \sum t_{i1} - \cdots - \frac{\lambda\lambda_m}{k_1} \sum t_{im} = Q_i \qquad (1)$$

$i = 1, 2, \cdots v$

where Σt_{ij} is the sum of the treatments which are j-th associate with the i-th treatment $ij = 1, 2, \cdots m$ and Q_i is the adjusted treatment total for the i-th treatment. Also,

$$Q_s = \Sigma_i Q_s{}^i$$

the summation is over the r values of i. A solution to these equations with the restriction

$$\Sigma t_i = 0$$

will give us the estimated treatment effects from the final design considering it as a single P.B.I.B. design.

If we analyse each B.I.B. experiment separately and estimate $t_j{}^i$ separately and substitute in the primary P.B.I.B. design, then the normal equation leading to the estimates of the treatment effects will be

$$\left(r - \frac{r}{k}\right) t_j - \frac{\lambda_1}{k} \sum t_{j1} - \cdots - \frac{\lambda_m}{k} \sum t_{jm} = P_j \qquad (2)$$

$i = 1, 2, \cdots v$

where

$$\text{(i) } P_j = \sum_i \hat{t}_j{}^i = \frac{Q_j}{r_1 E_1}$$

as the block totals are Zero by virtue of the restriction in each of the designs, and

$$\text{(ii) } r_1 E_1 = r_1 - \frac{r_1}{k_1} + \frac{\lambda_1}{k_1}.$$

Equation (2) reduces to

$$\left(rr_1 - \frac{rr_1}{k_1}\right) t_i - \frac{\lambda\lambda_1}{k_1} \sum t_{j1} - \cdots - \frac{\lambda\lambda_m}{k_1} \sum t_{jm} = Q_i \tag{3}$$

the solution of which is identical with the solution at (1) obtained through the other method. The method of solution reduces much simpler when the primary design is a B.I.B. design in which case the final design is also a B.I.B. design.

As a special case of the P.B.I.B. design, let us take the circular design as defined by Das (1960). The definition as given by Das is as follows. Let there be n equal arcs on the circumference of a circle denoted in order by $a_1, a_2, \cdots a_n$. If we form bigger arcs of size, A_{ki} such that it is the sum of k consecutive small arcs starting with a_i, we shall have in all n such arcs for n different values of i. Now, if we identify a_i with a set (si) of m treatments such that the different sets (si) are mutually exclusive, then the contents of the n acrs A_{ki}, will form an incomplete block design with n blocks, mn treatments, mk block size and k replications.

If the individual treatments be denoted by $t_{ij} (i = 1, 2, \cdots n; i = 1, 2, \cdots m)$ and the set of all the m treatments on the arc, a_i, be denoted (a_{ij}) the block contents of the design come out as below:

Block numbers	Treatments
1	$(a_{1j})\ (a_{2j}) \cdots (a_{kj})$
2	$(a_{2j})\ (a_{3j}) \cdots (a_{k+1,\ j})$
:	
n	$(a_{nj})\ (a_{1j}) \cdots (a_{k-1,\ j})$

Such designs can be obtained for any number of treatments and block sizes. In this design there are km treatments in each of the blocks. With this km treatments we form a B.I.B. design with parameters $(km, k_1, r_1, b, \lambda)$. This we repeat for all the blocks of the circular design. The final design will be a P.B.I.B. design but can be analysed just like circular design by effecting reduction of the number of normal equations as has been shown by Das (1960). The final design will have $d - 1$ associate classes where $d = [n/2]$.

2. *Incomplete Block Designs Obtained through Disconnected Designs*

If again we take a disconnected design of block size $(v - c)$ in g blocks, each treatment being replicated only once and connect these blocks by adding to each block a set of c treatments, the resultant design will have g blocks each of size v, there being c treatments common to all blocks. If now we form a B.I.B. design with parameters (v, k, r, b, λ) out of the treatments in each block, then the final design will be the same as the design given by Pavate (1961) as a group of B.I.B. designs with common treatments. He has analysed this design as usual. But this design can also be analysed in two stages as shown below.

With usual notations, the solution of the j-th treatment from i-th B.I.B. experiment is

$$t_j^i = \frac{Q_j^i}{\left(r - \frac{r}{k} + \frac{\lambda}{k}\right)} \tag{4}$$

and the set of normal equations from the primary design for the common treatments $j = 1, 2, \cdots c$ is

$$gt_j - \frac{g}{v}\sum^{c} t_i - \frac{1}{v}\sum\sum t_j^i = P_j \tag{5}$$

and that for the other treatments is

$$\left(1 - \frac{1}{v}\right)t_j^i - \frac{1}{v}\sum^{c} t_l - \frac{1}{v}\sum_{k \neq j}^{v-c} t_k^i = P_j^i \tag{6}$$

where P_j = adjusted treatment totals

$$= \sum^{g} t_j^i = \frac{\Sigma Q_j^i}{r - \frac{r}{k} + \frac{\lambda}{k}} \tag{7}$$

as all the block totals of the primary design will be Zero by Virtue of the restriction, *viz.*, $\sum_j t_j^l = 0$ taken while solving the normal equations in the l-th secondary design, and

$$P_j^i = t_j^i = \frac{Q_j^i}{r - \frac{r}{k} + \frac{\lambda}{k}}. \tag{8}$$

Hence

$$t_j = \frac{Q_j}{g\left(r - \frac{r}{k} + \frac{\lambda}{k}\right)} \tag{9}$$

$j = 1, 2, \cdots c$ for the common treatments subject to the condition

$$g \sum^{c} t_i + \Sigma\Sigma t_j^i = 0$$

and the solution of Equation (6) reduces to

$$t_j^i = \frac{Q_j^i}{rE} + \frac{1}{gcrE} \sum^{c} Q_i + \frac{1}{crE} \sum_{j=1}^{v-c} Q_j^i \tag{10}$$

which is identical with the solution given by Pavate (1961). Some more details regarding the analysis of such designs have been discussed by the authors elsewhere.

3. *Designs Obtainable through G.D. Designs in the Second Stage*

In the previous sections we developed a method of construction of incomplete block designs by combining two designs, the secondary design being always a B.I.B. design. As such designs require more number of replications, the possibility of using P.B.I.B. designs as secondary designs so that the number of replications may be reduced, was investigated. While it was found that P.B.I.B. design in general cannot be used as secondary designs, two particular cases of P.B.I.B. designs, *viz.*, G.D. designs and simple lattice designs can be applied to achieve the objective.

In a P.B.I.B. design with s associate classes and parameters say

$$(v, k, r, b, \lambda_1, \lambda_2, \cdots \lambda_s, n_1, n_2, \cdots n_s, p_{jk}^i, ijk = 1, 2 \cdots s)$$

m elements are associated to each element of this design and then a G.D. design with k groups of m treatments each is formed out of each block, then the final design will be an extended G.D. design with parameters, $v^* = vm$, $k^* = k_1$, $r^* = rr_1$, $b^* = bb_1$, $\lambda_1^* - \lambda_1\mu$, $\lambda_2^* = \lambda_1\mu_2$ $\cdots \lambda^*_{s+1} = \lambda_s\mu_2$, $n_1^* = m - 1$, $n_2^* = n_1 m$, $\cdots n^*_{s+1} = n_s m$ and the secondary parameters come out as below:

$$P' = \begin{bmatrix} m-2 & 0 & 0 & .. & ... & 0 \\ 0 & n_1 m & 0 & .. & ... & 0 \\ 0 & 0 & n_2 m & .. & ... & 0 \\ .. & .. & .. & .. & ... & ... \\ 0 & 0 & 0 & 0 & .. & n_s m \end{bmatrix}$$

$$p^{i+1} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & m-1 & 0 & \cdots & \cdots & 0 \\ 0 & p_{11}{}^i m & \cdots & \cdots & p_{1i}{}^i m & \cdots & \cdots & \cdots & p_{1s}{}^i m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m-1 & p_{11}{}^i & \cdots & \cdots & p_{ii}{}^i m & \cdots & \cdots & \cdots & p_{ss}{}^i m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & p_{s1}{}^i m & \cdots & \cdots & p_{si}{}^i m & \cdots & \cdots & \cdots & p_{ss}{}^i m \end{bmatrix}$$

$i = 1, 2, \cdots m.$

Analysis.—With usual notations the normal equations giving the solution of t_{ij} are given by

$$\left(rr_1 - \frac{rr_1}{k_1} + \frac{r\mu_1}{k_1}\right) t_{ij} - \frac{r\mu_1}{k_1} \sum_j t_{ij} - \frac{\lambda_1\mu_2}{k_1} \sum_1 \sum_j t_{ij} - \cdots$$

$$\cdots - \frac{\lambda_s\mu_2}{k_1} \sum_s \sum_j t_{ij} = Q_{ij} \tag{11}$$

$i = 1, 2, \cdots v; \; j = 1, 2, \cdots m$

where Σ_1 denotes the sum over the treatments which are 1st associates with the treatment in question and a similar definition for $\Sigma_2, \cdots \Sigma_s$ and Q_{ij} denoting the adjusted treatment total for the *ij*-th treatment.

Summing the set of equations (11) over $j = 1, 2, \cdots m$, we have

$$\left(rr_1 - \frac{rr_1}{k_1} + \frac{r\mu_1}{k_1} - \frac{rm\mu_1}{k_1}\right) G_i - \frac{m\lambda_1\mu_2}{k_1} \sum G_{i1} - \cdots$$

$$\cdots - \frac{m\lambda_2\mu_2}{k_1} \sum G_{i3} = \sum_j Q_{ij} = p_i \tag{12}$$

$i = 1, 2, \cdots v$

where

$$G_i = \sum_j t_{ij} \quad \text{and} \quad \Sigma G_{i1} = \Sigma_1 \Sigma_j t_{ij}.$$

This set of v equations are similar to the set of v normal equations of the initial P.B.I.B. design. Hence from these equations we can solve for G_i, the solution being similar to that of the initial P.B.I.B. design. Now from (11) we have

$$\left(rr_1 - \frac{rr_1}{k_1} + \frac{r\mu_1}{k_1}\right) t_{ij} = Q_{ij} - \frac{P_i}{m} + \left(rr_1 - \frac{rr_1}{k_1} + \frac{r\mu_1}{k_1}\right) \frac{G_i}{m} \tag{13}$$

Substituting the value of G_i in (13) we have the solution.

The adjusted S.S. can be now obtained from

$$\Sigma t_{ii} Q_{ij}$$

If, however, we take a B.I.B. as the initial design, the analysis is simplified still further.

4. *Designs Obtained through Simple Lattice in the Second Stage*

In the last section we have considered a G.D. design in the second stage which has the advantage that designs with smaller number of replications than that in a corresponding B.I.B. design can be obtained by using them. In order to reduce the number of replications still further, we consider another class of P.B.I.B. designs, *viz.*, lattice design to be used as the secondary design. For simplicity let us consider only simple lattice which has two replications. These series of designs will have smaller number of replications.

In a P.B.I.B. design with m associate classes and parameters, say

$$(v, k, r, b, \lambda_1, \lambda_2 \cdots \lambda_m, n_1, n_2, \cdots n_m \text{ and } q^i_{jk},\ ijk = 1, 2 \cdots m)$$

jk treatments are associated to each element of this design and a simple lattice design out of the treatments of each block of the primary design is formed. The final design so obtained will be a P.B.I.B. design with $2m+1$ associate classes but there will be only $m+1$ distinct values of λ, the remaining λ's being zero. The parameters of the final design will be $v^* = vk$, $k^* = k$, $r^* = 2r$, $b^* = 2bk$, $\lambda_1{}^* = r$, $\lambda_2{}^* = \lambda_1, \cdots$ $\lambda^*{}_{m+1} = \lambda_m$, $\lambda^*{}_{m+i} = 0\ (i = 2, \cdots m)$, $n_1{}^* = k-1$, $n_2{}^* = n_1, \cdots$ $n^*{}_{m+1} = n_m$, $n^*{}_{m+2} = n_1\ (k-1) \cdots n^*{}_{2m+1} = n_m\ (k-1)$ and the secondary parameters come out as below:

$$P^1 = \begin{bmatrix} k-2 & 0'(1\times m) & 0'(1\times m) \\ 0(m\times 1) & 0(m\times m) & D(n_1, n_2, \cdots n_m) \\ 0(m\times 1) & D(n_1, n_2 \cdots n_m) & (k-2)\, D(n_1, \cdots n_m) \end{bmatrix}$$

Where $0'(1\times m)$ is an $1\times m$ matrix with all Zero elements and $D(n_1, \cdots n_m)$ is an $m\times m$ matrix whose diagonal elements are $n_1, n_2, \cdots n_m$ and other elements are zero.

$$P^{i+1} = \begin{bmatrix} 0 & 0'(1\times m) & (k-1)\, e_i' \\ 0(m\times 1) & Q_i(m\times m) & 0(m\times m) \\ (k-1)\, e_i & 0(m\times m) & (k-1)\, Q_i(m\times m) \end{bmatrix}$$

$i = 1, 2, \cdots m$.

Where e_i' is $1\times m$ matrix, the i-th element being unity and other elements are Zero and Q_i $(m\times m)$ is the $m\times m$ matrix corresponding to the i-th associates in the initial P.B.I.B. design *viz.*, (q_{ik}^{i}) and

$$P^{j+m+1} = \begin{bmatrix} 0 & e_j' & (k-2)\, e_j' \\ e_j & 0\,(m\times m) & Q_i\,(m\times m) \\ (k-2)\, e_j & Q_j\,(m\times m) & (k-2)\, Q_j \end{bmatrix}$$

$= 1,\ 2, \cdots m.$

Though the design appears to be a P.B.I.B. design with $2m+1$ associate classes the analysis can be simplified to that of the original P.B.I.B. design.

Intra-block analysis.—From the usual postulates the intra-block estimate of any linear treatment contrast is obtained by substituting in the contrast the solution of the normal equations,

$$\left(2r - \frac{2r}{k} + \frac{r}{k}\right) t_{ij} - \frac{r}{k} \sum_j t_{ij} - \frac{\lambda_1}{k} \sum t_j^{i1} - \frac{\lambda_2}{k} \sum t_j^{i2} \cdots$$

$$\cdots - \frac{\lambda_m}{k} \sum t_j^{im} = Q_{ij} \qquad (14)$$

$i = 1,\ 2 \cdots v;\quad j = 1,\ 2 \cdots k$

where t_{ij} is the j-th treatment in the i-th group. $\Sigma\, t_j^{i1}$ is the sum of the treatments which occur with ij-th treatment in λ_1 of the blocks and similar definitions for $\Sigma\, t_j^{i2} \cdots$ and Q_{ij} is the adjusted treatment total.

If we denote $\sum_j t_{ij}$ as G_i, we have, by summing (14) over all j for a given i.

$$\left(r - \frac{r}{k}\right) G_i - \frac{\lambda_1}{k} \sum G^{i1} - \frac{\lambda_2}{k} \sum G^{i2} - \cdots - \frac{\lambda_m}{k} \sum G^{in}$$

$$= \sum_j Q_{ij} = Q_i \qquad (15)$$

$i = 1,\ 2, \cdots v.$

These equations are all not independent but only $(v - 1)$ of them are independent. Hence with the condition $\sum_i G_i = 0$ we can solve for G_i from (15), the solution will be similar to that of a general P.B.I.B. design with m associate classes.

Equation (14) can now be written as

$$\left(2r - \frac{r}{k}\right) t_{ij} - \frac{\lambda_1}{k} \sum t_j^{i1} - \frac{\lambda_2}{k} \sum t_j^{i2} - \cdots - \frac{\lambda_m}{k} \sum t_j^{im}$$

$$= Q_{ij} + \frac{r}{k} G_i = P_{ij} \tag{16}$$

$i = 1,\ 2,\ \cdots v;\ j = 1,\ 2, \cdots k.$

Summing Equation (16) over all i for a given j we have on further simplification

$$\sum_i t_{ij} = \frac{\sum_i Q_{ij}}{r} \tag{17}$$

$j = 1,\ 2 \cdots k.$

Equation (16) together with (17) give the solution of

$t_{ij},\ i = 1,\ 2, \cdots v,\ j = 1,\ 2, \cdots k.$

The analysis is much simpler in the case where the initial design is a B.I.B. design.

5. *By Association of Incomplete Block Designs*

In this section a method of construction of another series of incomplete block designs has been described by combining two incomplete block designs, but in a manner different from the method of combining two designs described earlier.

In a P.B.I.B. design with parameters $(v, b, r, k, \lambda_1, \lambda_2, \cdots \lambda_m, n_1, n_2, \cdots n_m, p_{ij}^k, i, j, k = 1, 2, \cdots m)$ say, associate s numbers $(a_1, a_2, \cdots a_s)$ to each of the elements, so that in a block there will be ks numbers, each denoting a treatment. In the method described earlier, an incomplete block design was formed with these ks treatments, by using B.I.B., G.D. and Simple Lattice designs, as the secondary designs. Now instead of forming the second design with these ks treatments as such, let us group the ks treatments into s groups each containing k treatments, such that all the k treatments containing, say, a_i will form the i-th group $(i = 1, 2, \cdots s)$. With these s groups form a P.B.I.B. design with n associate classes, considering each group as a single treatment. If we repeat this for all the blocks then the final design will be a P.B.I.B. design with $(m+1)(n+1)-1$ associate classes, the parameters for which have been indicated later.

As a special case if we take one of the designs to be a P.B.I.B. design with m associate classes and the other a B.I.B. design, then the final design will be a P.B.I.B. design with $2m+1$ associate classes.

Again, if both the designs are B.I.B. designs the final design is a P.B.I.B. design with 3 associate classes.

In general if the two P.B.I.B. designs are (v_1, k_1, r_1, b_1, λ_1, λ_2, $\cdots$ λ_m, n_1, n_2, $\cdots$ n_m, p_{jk}^i; $ijk = 1, 2, \cdots m$ and v_1^*, k_1^*, r_1^*, b_1^*, λ_1^*, λ_2^*, $\cdots$ λ_l^*, $n_1^* n_2^*$, $\cdots$ n_l^*, q_{jk}^i; $ijk = 1, 2, \cdots l$) then the final design is a P.B.I.B. design with the following parameters.

$$v_1 v_1^*,\ k_1 k_1^*,\ r_1 r_1^*,\ b_1 b_1^* \text{ with } \lambda \text{ value } r_1\lambda_i^* \quad i = 1, 2, \cdots l,$$

$$\left.\begin{matrix} \lambda_j r_1^* \\ \lambda_j \lambda_i^* \end{matrix}\right\} \quad i = 1, 2, \cdots l; \quad j = 1, 2, \cdots m$$

with the n values

$$n_i^* \qquad i = 1, 2, \cdots l$$

$$\left.\begin{matrix} n_j \\ n_i^* n_j \end{matrix}\right\} \quad i = 1, 2, \cdots l; \quad j = 1, 2. \cdots m \text{ respectively.}$$

The secondary parameters have also been worked out and is as follows:

$$P^i = \begin{bmatrix} Q_i & F' \\ F & D \otimes Q_{irc} \end{bmatrix} \qquad i = 1, 2 \cdots l$$

where $F = m \times 1$ column matrix in which each of the element is $0\,[(l+1) \times l]$ where $0\,(1 \times l)$ is a $1 \times l$ matrix each with all Zero elements and

$$Q_{irc} = \begin{bmatrix} 0 & e_i' \\ e_i & Q_i \end{bmatrix}$$

where e_i is $1 \times l$ matrix, the i-th element being unity and other elements are Zero and $Q_i(l \times l)$ is the $l \times l$ matrix corresponding to the i-th associates of the second P.B.I.B. design, *viz.*, $((q_{jk}^i))$ and D is the diagonal matrix in which the diagonal elements are n_1^*, n_2^*, $\cdots$ n_r^* and $\otimes$ denotes Kronecker product.

$$P^{l+j} = \begin{bmatrix} 0\,(l \times 1) & G_j' \\ G_j & P_j \otimes D_{rc} \end{bmatrix}$$

where G_j is the $m \times 1$ column matrix in which the j-th element is Dr and other elements are 0 $[(l \times 1) \times l]$ where

$$D_{rc} = \begin{bmatrix} 1 & 0\,(l \times l) \\ 0\,(l \times 1) & D \end{bmatrix} \quad \text{and} \quad Dr = \begin{bmatrix} 0\,(1 \times l) \\ D \end{bmatrix}$$

and P_j is the $(m \times m)$ matrix corresponding to the j-th associate in the first P.B.I.B. design, *viz.*, $P_j = ((p_{rs}^{i}))$ and

$$P^{l+j+ij} = \begin{bmatrix} 0\,(l \times l) & F_j \\ F_j & P_j \otimes Q_{irc} \end{bmatrix} \qquad \begin{matrix} i = 1,\ 2, \cdots l \\ j = 1,\ 2, \cdots m \end{matrix}$$

where F_j is the $(m \times 1)$ column matrix in which the j-th element is Q_{ir} and other elements are 0 $[(l + 1) \times l]$

$$Q_{ir} = \begin{bmatrix} e_i' \\ Q_i \end{bmatrix}.$$

The proof of the parameters v, r, k and b follow immediately. Denoting the treatments of the first design by a, b, $\cdots$ and that of the second design by α, β, $\cdots$, the proof of the occurrence of λ values as indicated can be shown as follows: Suppose a and b are i-th associates, *i.e.*, they occur together in λ_i blocks of the first design then the treatment pair $a\alpha$, $b\alpha$ occur in $\lambda_i r_1^*$ blocks of the final design since α occurs r_1^* times corresponding to each block of the first design $i = 1, 2, \cdots m$. Similarly $a\alpha$, $a\beta$ occur in $r_1\lambda_j^*$, $j = i, 2, \cdots l$ blocks of the final design where α and β are j-th associates in the second design, and $a\alpha$, $b\beta$ occur in $\lambda_i\lambda_j^*$ $i = 1, 2, \cdots m$, $j = 1, 2, \cdots l$ blocks of the final design where a and b are i-th associates in the first design and α and β are j-th associates in the second design. The secondary parameters can also be proved by argument.

These results agree with that of Shah (1960) who, using a different method, got the same designs.

6. *Applications to the Construction of Factorial Experiments*

The method explained above can be used for the construction of confounded factorial designs as follows:

To construct the design $a \times b \times c \times d$ where a, b, c and d are respectively the number of levels of factors which will be denoted as

X, B, C and D, let us take a suitable incomplete block design with 'a' as the number of treatments and k as block size and associating $b \times c \times d$ combinations of 3 factors to each treatment of the incomplete block design, we can group the $k \times b \times c \times d$ in each block into $b \times c \times d$ groups, such that the combinations in any group contain one combination only of the factors B, C and D and the levels of the factor x varies over the K values corresponding to the block of the incomplete block design. With $b \times c \times d$ groups if we form a balanced factorial experiment with convenient block size and if this is repeated for all blocks then the final design is also a factorial design. If the initial incomplete block design is a balanced one then the final design will be a balanced factorial experiment and if the initial incomplete block design is a partially balanced design then the final design will be a partially balanced factorial experiment.

The interaction confounded in the second design will be confounded in the final design also and its loss of information will be the same. The main effect of the first factor, which is in the form of a B.I.B. design is also partially affected by the block effects and its loss of information will be the same as that of the loss of information in the corresponding B.I.B. design. The interactions which are the product of the above two sets of interactions are also affected and the loss of information will be the product of the losses of information of the corresponding two interactions of the above two sets.

SUMMARY

Three methods of construction of incomplete block designs by associating two incomplete block designs have been presented in this paper. One method consists in first taking an incomplete block design (called primary design) and then forming a B.I.B. design (called the secondary design) with the block contents of each of the blocks of the primary design. All these secondary designs obtained thus give another incomplete block design when put together. Utilising B.I.B., P.B.I.B., etc., designs as primary designs, several series of designs have been obtained through this method. All the designs obtainable through this method can be analysed in two stages also in addition to the ordinary method. First stage of analysis consists of analyses of each of the secondary designs separately. The observations in the blocks of the primary designs are then taken to be these estimates, each design supplying the contents of the corresponding block. An ordinary analysis of the data of the primary design thus obtained, will give the true estimates of the treatment effects.

The second method of the construction is similar but for the fact that the secondary design formed out of the block contents of the primary design is either a simple lattice or a group divisible design. A method of intra-block analysis for all designs obtained through these methods has been presented. If the primary design is a P.B.I.B. design with m associate classes and the secondary design, a simple lattice, then the final design will be a P.B.I.B. design with $2m + 1$ associate classes, but the analysis can be simplified to that of the primary P.B.I.B. design.

The third method of construction consists in combining two designs but in a different way. In this method any incomplete block design is first taken and then some, s numbers are associated with each of the treatments in the incomplete block design. Next the contents of each block are made into groups each containing all the treatments containing a particular associated number. Another incomplete block design is then formed taking these groups as treatments to get a new series of P.B.I.B. designs.

ACKNOWLEDGEMENTS

We are highly thankful to Dr. G. R. Seth and Dr. V. G. Panse, Statistical Adviser, Indian Council of Agricultural Research, for providing us with adequate facilities for our investigation.

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